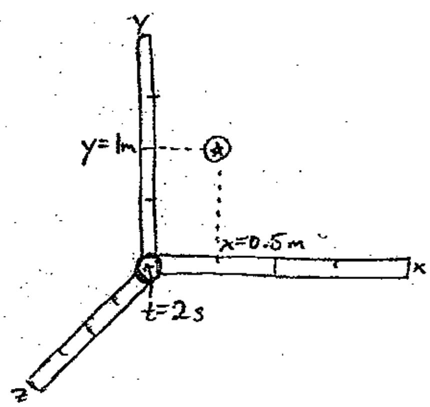
Name: Group Members:

Physics 200 Tutorial 3:

The Lorentz Transformation

An EVENT is something that happens at a particular place and a particular time, like a cell-phone ringing or two protons colliding at the Large Hadron Collider. Observers in any frame of reference can assign coordinates and times to any given event by measuring the location and time of the event with their rulers and clocks:

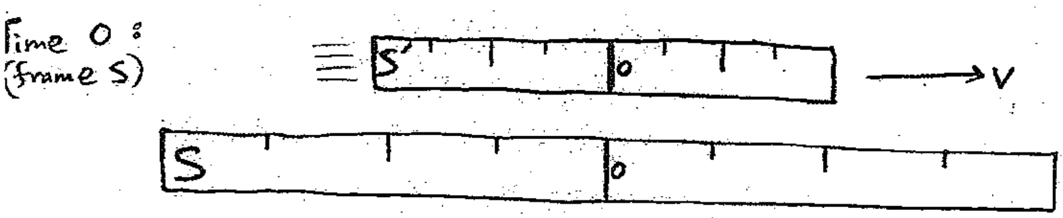


$\rightarrow (t, x, y, z) = (2s, 0.5m, 1m, 0)$

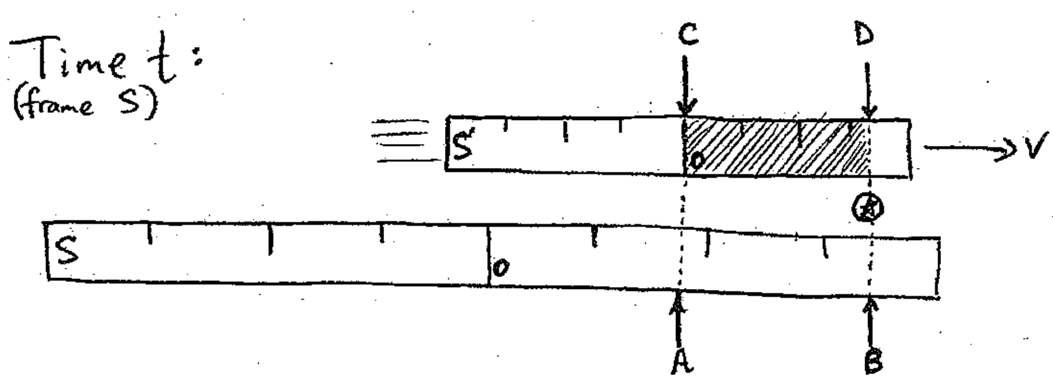
Observers in different frames of reference will generally measure different coordinates and times for the same event. The LORENTZ TRANFORMATION gives the precise relation between the coordinates and times as measured by one observer, and the coordinates and times as measured by another observer moving at a constant velocity relative to the first observer. In this tutorial, you will derive the Lorentz transformation and get a chance to see how useful it is.

We begin by supposing there are two frames of reference, which we'll call S and S'. We will assume that an event that occurs at $(t,x,y,z) = (0,0,0,0)$ in frame S is also defined to be at coordinates $(t',x',y',z') = (0,0,0,0)$ in frame S'. This is possible since the observers in frame S' are free to pick where they want their origin of coordinates to be and when to set their clocks to 0. We will also assume that the observers in the two frames agree on which direction is $\hat{x}$, which is $\hat{y}$, and which is $\hat{z}$. In particular, they call $\hat{x}$ the direction of their relative motion, so observers in frame S observe frame S' to move at velocity $v\hat{x}$ and observers in frame S' observe frame S to move at velocity $-v\hat{x}$.

Here's the picture at time t=0 in frame S (we'll ignore the y and z coordinates for now):



We have drawn the two rulers used by observers in frames S and S' to measure x positions. The rulers are identical, but we have taken into account length contraction when drawing the picture. Below is the picture at some time t in frame S, when some event (marked by ⊛) occurs at position x along the S ruler.



Question 1

a) What is the distance between A and B as measured along the lower ruler (answer in terms of v,x, and t; ignore the markings on the ruler)?

Position of A:

Position of B:

Distance B – A:

b) This distance is the observed length of the shaded region on the upper ruler. What is the proper length of this shaded region?

Answer:

c) The coordinate x' that the observers in frame S' assign to the event is the distance indicated on their ruler at location D. This is:

$x' =$

↑ answer in terms of x, v, t, γ

The formula you have derived is the first part of the Lorentz transformation. It gives the coordinate x' of an event in frame S' in terms of the coordinate x and time t of the event in frame S, and the velocity v of frame S' relative to frame S.

d) Write a formula for the coordinate x of an event in frame S in terms of the coordinate x' and time t' of the event in frame S', and the velocity v' of frame S relative to frame S'. *(Hint: is this any different from part c), other than the labels?)*

x =

e) What is the velocity v' of frame S relative to frame S'?

v' =

f) Using this and your answers to part d) and e), write a formula for x in terms of x',t', and v.

x =

Check your answer with a TA

g) Using your results for c) and f), solve for t' in terms of x,t,v,c, and γ.

t' =

(hint: it is helpful to pull out a factor of γ and write t' = γ (...). Then you can simplify what's in the brackets using $1/\gamma^2 = 1 - v^2/c^2$).

h) What are y' and z' in terms of v, c, x, y, z, and t?

y' =

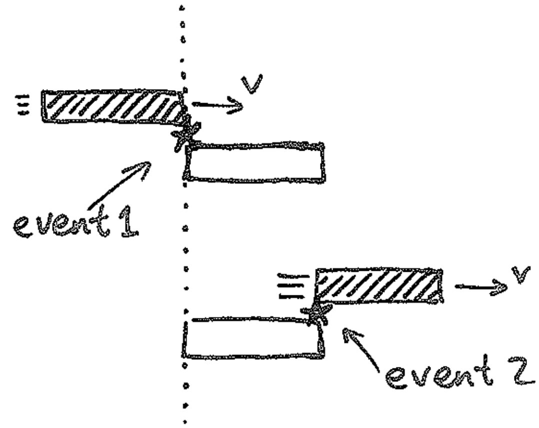
z' =

Your answers to c), g), and h) give the full set of coordinates and time measured by observers in frame S' in terms of the coordinates and time measured in the frame S. You have just derived the Lorentz Transformation!

Question 2

As an example of how to use the Lorentz transformation, consider the third homework problem:

A train of length 300m observes another train on a parallel track coming towards it at $v = \sqrt{3/4}\,c$. The other train appears (i.e. is measured by the first train) to have length 300m also. In the reference frame of the second train, how long does it take for the two trains to pass each other (i.e. what is the time between when the fronts align and when the backs align)



This problem is asking for the time between two events in the frame of the moving train. As a first step, what are the times and positions of these two events in the original frame?

Frame of stationary train:

we can choose our coordinates so this is true

EVENT 1: time $t_1 = 0$

position $x_1 = 0$

EVENT 2: time $t_2 =$

position $x_2 =$

Now, using the Lorentz transformation, determine the times for the two events in the frame of the moving train

Frame of moving train:

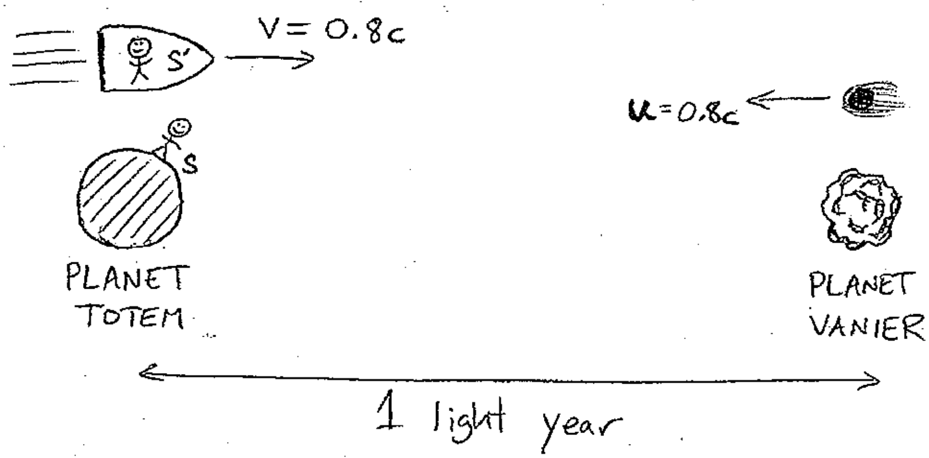
EVENT 1: time $t'_1 =$

EVENT 2: time $t'_2 =$

In the frame of the moving train, how long does it take for the two trains to pass each other?

Problem 3

Many of the formulae we have derived break down if we try to use a relative velocity greater than the speed of light. I mentioned in class that nothing can actually travel faster than the speed of light. But what if we had two objects coming towards each other, both travelling at nearly the speed of light? Wouldn't their relative velocity then be greater than the speed of light? In this question, we'll use the Lorentz transformation to analyze this.



In the picture above, two planets are separated by 1 light year. At time t=0 in the frame of the planets (let's call this frame S), a spaceship passes Planet Totem at v=4/5c travelling toward Planet Vanier, and a comet passes Planet Vanier at velocity u=4/5c travelling towards planet Totem. We would like to determine what velocity observers on the ship measure for the comet. To determine the velocity, these observers measure the positions and times when the comet passes planet Vanier and Planet Totem and then plug everything in to good-old $\Delta x/\Delta t$ to determine the velocity. In order to determine the positions and times that observers in the ship's frame measure for these two events, we'll first determine the positions and times in the frame of the planets and then apply the Lorentz Transformation.

Assume that observers in frame S define the event where the ship passes Planet Totem to be at x=0 and t=0.

a) For the event where the comet passes Planet Vanier, what is the position and time as measured in the frame of the planets (frame S)?

$x_V =$

$t_V =$

b) For the event where the comet passes Planet Totem, what is the position and time as measured in the frame of the planets (frame S)?

$x_T =$

$t_T =$

c) Using the Lorentz Transformation, determine the positions and times for these events as measured in the frame of the ship. Remember that the v and γ in the Lorentz Transformation refer to the velocity of the frame S' relative to the frame S.

$x'_V =$

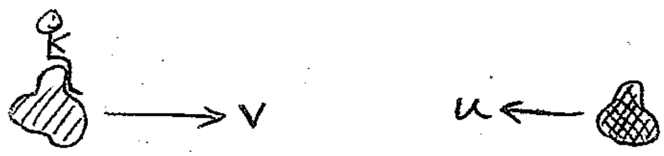
$t'_V =$

$x'_T =$

$t'_T =$

d) What speed do observers on the ship measure for the comet? Is this greater than c?

$u' = (x'_T - x'_V)/(t'_T - t'_V) =$



e) Derive a general formula for the relative velocity of two objects if the objects are observed to be travelling towards each other with speed v and u respectively. What does your formula give if we take u to be c?