Physics 200 Problem Set 8

You may wish to read through the notes on expectation values and uncertainty on the course webpage before doing this assignment.

Problem 1

The wavefunction for an electron in a short thin wire is

$$\psi(x) = Ax \qquad 0 \le x \le 10nm$$

with $\psi(x) = 0$ everywhere else.

a) Sketch the wavefunction and probability density for this electron.

b) What must the constant A be?

c) If the position of the electron is measured, what is the probability that it will be found at $x = 5nm$?

d) If the position of the electron is measured, what is the probability that it will be found in the range $0 \le x \le 3nm$?

e) If we did a position measurement many times with this same initial wavefunction, what would be the average value of all our measurements? In other words, what is the EXPECTATION VALUE of x for this wavefunction?

Problem 2

In class, we said that the wavefunction for an electron with momentum p should have wavelength $h/|p|$. But this is the same wavelength for p and $-p$. How does the wavefunction differ for an electron moving to the right vs an electron moving to the left? Putting it another way, the function $cos(\frac{2\pi p}{h}x)$ is the same for p and $-p$, so what tells us the direction of motion?

This is one place where it is essential to remember that the wavefunction is allowed to be complex. The function $cos(\frac{2\pi p}{h}x)$ is the real part of either $e^{i\frac{2\pi p}{h}x}$ or $e^{-i\frac{2\pi p}{h}x}$. It turns out that the $e^{i\frac{2\pi p}{h}x}$ is the wavefunction for a particle with momentum p while $e^{-i\frac{2\pi p}{h}x}$ is the wavefunction for a particle with momentum $-p$. In this question, we'll be able to understand why.

For an ordinary wave, the time-dependent traveling wave is described by a function of the form $cos(kx - \omega t)$. The complex version of this is

$$\psi(x,t) = e^{i(kx-\omega t)}$$

This is the time-dependent wavefunction for a traveling electron with momentum p if we take $k = \frac{2\pi p}{h}$ (ω is also determined in terms of p, as we will discuss in class).

a) Assume that we have chosen units so that $k = 1$ and $\omega = 1$. Plot the real part of the wavefunction at time $t = 0$ (solid line) and at time $t = \pi/4$ (dotted line) on the same axes.

b) Now repeat part a) but choose $k = -1$ (i.e. negative momentum) and $\omega = 1$. How does the time dependence of the wave differ from the one in part a?

c) Plot the probability density for an electron with the wavefunction above. Is it wavy? *As we said in class, this wavefunction isn't really physical, since it can't be normalized. For a real electron, the wavepacket would always have a finite width and therefore be a superposition of these complex waves.*

Problem 3

The wavefunction for an electron in a short thin wire is

$$\psi(x) = \frac{1}{\sqrt{a}}e^{-|x|/a} \qquad -\infty \le x \le \infty$$

where $a = 1nm$

a) If we performed a measurement of position on 1000 electrons with this same initial wavefunction, how many would we expect to find with $x > 5nm$?

b) Suppose we now want to make a prediction about the what will happen if we measure momentum. We then need to write this wavefunction as a combination of momentum eigenstates, i.e. pure waves $e^{i\frac{2\pi p}{h}x}$ with various momenta p.

Formally this sum can be written as

$$\psi(x) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} dp A(p) e^{i\frac{2\pi p}{h}x} \ .$$

where the integral sums over all possible momenta p, and the momentum wavefunction $A(p)$ tells us how much of each wave we have. The Fourier Transform formula tells us that $A(p)$ is given by

$$A(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} dx \psi(x) e^{-i\frac{2\pi p}{h}x} \ .$$

Using this formula, show that the momentum wavefunction for our electron with wavefunction $\psi(x)$ (given at the beginning) is

$$A(p) = 2\sqrt{\frac{a}{h}} \frac{1}{1 + \frac{4\pi^2 a^2}{h^2} p^2}$$

Hint: You will need to break up the integral in the $A(p)$ formula into two pieces. The formula $\int_0^\infty e^{-ax} dx = 1/a$ still works when a is a complex number, as long as its real part is positive.

c) Now let's apply our result. If we measure the velocity of the electron with wavefunction $\psi(x)$ (given at the beginning of the question), what is the probability that we'll find it's velocity to be larger than $10^6 m/s$ (in the positive direction)?

Problem 4

a) The precise definition of uncertainty is the standard deviation of the probability density (when viewed as a distribution of data), which measures how far on average each data point is from the average value (standard deviation is 0 when all the data points have the same value). When the average value is 0 (as for the probability distribution of the wavefunction in question 3), the definition of standard deviation simplifies to

$$(\Delta x)^2 = \int_{-\infty}^{\infty} dx P(x) x^2$$

i.e. we take the average value of x^2 and then take the square root. Using this, calculate the uncertainty Δx for the particle with wavefunction $\psi(x)$ in question 3. (Feel free to use a calculator/computer to do the integral)

b) Using the same method, calculate the uncertainty Δp in momentum. c) How does the product $\Delta x \Delta p$ compare with Heisenberg's minimum $h/(4\pi)$?