Basic Probability

There are many situations in which a certain operation (e.g. flipping a coin, rolling a die) is performed for which we know the possible outcomes (distinct possibilities for the results of the operation) but we have no way of predicting the outcome. On the other hand, there are predictions we can make about the whole collection of outcomes if we repeat the operation a large number of times.

For example, we know that a flipped coin will come up as heads about half the time and tails about half the time. To make this mathematically precise, we must use the idea of PROBABILITY. The idea is that if we were to flip the coin a very large number of times (say N) and count how many times we get heads (call this N_H) then the ratio N_H/N always approaches the same value (in this case 1/2) as N is taken to infinity. In this case, we say that the probability of finding heads is 1/2.

Similarly, when we roll a die, after a large number of rolls, the ratio between N_5, the number of times we roll 5, and N the total number of rolls, should be very close to 1/6, and should approach this value exactly as N goes to infinity. In this case, we say that the probability of rolling a 5 (or any other number) is 1/6.

These two examples are simple, since all the outcomes have an equal probability, but this doesn't have to be the case. We could have a coin that is heavier on one side so that tails comes up more often. Then we might have the probability of getting tails to be 2/3 and the probability of getting heads to be 1/3. The only constraint is that the sum of probabilities for all possible outcomes equals one.

It's important to note that if an outcome has a probability of 1/2, this doesn't mean that it is guaranteed to happen if we repeat the operation twice, or even ten times. For example, it's entirely possible to get ten heads in a row; we can only say that if you keep flipping the coin, the number of times heads comes up will tend to 1/2.

Probability of more general occurrences

So far, we've talked about the probability for an individual outcome of an operation. But sometimes, we want the probability of something more complicated, like rolling a die and getting a number that is a perfect square. In this case, there are two possible outcomes that match the description, either a 1 or a 4. To find the probability of rolling a perfect square, we just need to add up the probabilities for the individual outcomes that match our criterion. So in this case, we have

$$P_{square} = P_1 + P_4$$

In simple cases where the outcomes are all equally likely, the probability of some general result will just be the total number of outcomes that fit the criterion divided by the total number of outcomes. As an example, if we roll a red die and a blue die, there are 36 possible outcomes, and the probability that the numbers add up to 3 is just $2/36 = 1/18$, since there are two outcomes ($red = 1, blue = 2$ and $red = 2, blue = 1$) where the numbers add up to 3 and 36 total outcomes possible.

Probabilities for multiple independent events

Sometimes, we want to know the probability that two (or more) independent things both happen. For example, we want to know the probability that if we flip two coins, the first one will be heads and the second one will be tails. In this case, we just need to multiply the probabilities together:

$$P(A \; and \; B) = P(A) \times P(B) \tag{1}$$

Why is this true? If we think about doing this experiment a very large number of times, then a fraction $P(A)$ of the time, the first coin will be heads. If we think about all the times that the first coin is heads and ask what fraction of those times the second coin will be tails, the answer is just $P(B)$, since we are assuming the second coin flip is completely unaffected by the result of the first coin flip. Thus, overall, the fraction of times that the first coin is heads and the second coin is tails is $P(A)P(B)$ (which is one quarter in our example).

An example

As a final example, suppose we roll two dice and we want to know the probability that exactly one of the dice will be 1. To work this out, we need to add up the probabilities for all the specific things that could happen that match our desired outcome of getting 1 exactly once. We could have the first die being 1 and the second not being 1, or we could have the second die being 1 and the first not being 1. Each of these two cases is an example where we have multiple independent events, so we can calculate the probabilities using the formula (1). We have

$$P(first \; die \; 1 \; AND \; second \; die \; not \; 1) \;\; = \;\; P(first \; die \; 1) \times P(second \; die \; not \; 1)$$

$$\begin{aligned} &= \frac{1}{6} \times \frac{5}{6} \\ &= \frac{5}{36} \end{aligned}$$

and

$$\begin{aligned} P(first\ die\ not\ 1\ AND\ second\ die\ 1) &= P(first\ die\ not\ 1) \times P(second\ die\ 1) \\ &= \frac{5}{6} \times \frac{1}{6} \\ &= \frac{5}{36}\ . \end{aligned}$$

So finally

$$P(exactly\ one\ die\ being\ 1) = \frac{5}{36} + \frac{5}{36} = \frac{5}{18}\ .$$