Group Velocity and Phase Velocity

Light waves in the vacuum have the special property that the velocity is exactly the same for all wavelengths. This is not true for light propagating in materials (otherwise a prism wouldn't work) or most other kinds of waves, such waves in water. In general, waves with different wavelengths travel at different velocities.

Mathematically, this has to do with the relationship between frequency and wavelength. In general, pure waves propagating in the $+\hat{x}$ direction can be represented as

$$Ze^{i(kx-\omega t)}$$

(or the real part of this, if we are dealing with real functions). The *wave number* k is related to the wavelength by

$$k = \frac{2\pi}{\lambda} \ .$$

For a given type of wave, the frequency ω is determined in terms of k or alternatively, the wavelength. For example, with light waves, we have

$$\omega = ck$$

which means that the pure waves can be written as

$$Ze^{ik(x-ct)} \ .$$

Any superposition of these pure waves will always be some function of the form $f(x - ct)$, which means that as time increases, the function is simply translated to the right with velocity c. So clearly, the velocity of any wavepacket we build, like the velocity of the pure waves, is exactly c.

Now what happens if we have waves for which $\omega(k)$ is not simply equal to vk? In this case, the pure wave with a given k takes the form

$$Ze^{ik(x-\frac{\omega(k)}{k}t)}$$

so the pure wave with wave number k moves to the right with velocity

$$v_{pure}(k) = \frac{\omega(k)}{k} \ . \tag{1}$$

This velocity will be different for different ks, so it is a bit more complicated to understand how wavepackets made from these pure waves evolve with time.

To do this, let's consider a very long wavepacket with wavelength λ_0. This will be a superposition of pure waves with some narrow range of wave numbers near $k_0 = \frac{2\pi}{\lambda_0}$. Let's suppose the function $A(k)$ determines the amount of the pure wave with wave number k in the superposition (this is analogous to our $\tilde{\psi}(p)$). Then we can write the wavepacket as

$$h(x,t) = \int dk A(k) e^{i(kx - \omega(k)t)}$$

Since we are assuming that $A(k)$ is significant only over a narrow range of frequencies, it is a good approximation to say that in this range,

$$\omega(k) \approx \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k_0} (k - k_0)\ . \tag{2}$$

That is, we can approximate $\omega(k)$ by a straight line with the same value and same slope as $\omega(k)$ at $k = k_0$. Using this approximation, we have

$$\begin{aligned} h(x,t) &= \int dk A(k) e^{i(kx - \omega(k_0)t - \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0)t)} \\ &= e^{i(k_0 x - \omega(k_0)t)} \int dk A(k) e^{i(k-k_0)(x - \left.\frac{d\omega}{dk}\right|_{k_0} t)} \\ &= e^{i(k_0 x - \omega(k_0)t)} F(x - \left.\frac{d\omega}{dk}\right|_{k_0} t) \end{aligned}$$

We see that using our approximation, the wavepacket for a narrow range of frequencies splits up exactly into the product of the pure wave with wave number k_0 and the function

$$F(x - \left.\frac{d\omega}{dk}\right|_{k_0} t) = \int dk A(k) e^{i(k-k_0)(x - \left.\frac{d\omega}{dk}\right|_{k_0} t)}$$

Here, the pure wave factor is responsible for the "ripples" in the wavepacket, so the function F must be what defines the overall shape of the wavepacket (see accompanying figure).

We can see that the pure wave in the product moves to the right with velocity

$$v_{phase} = \frac{\omega(k_0)}{k_0}\ ,$$

exactly as in (1). This is known as the PHASE VELOCITY. On the other hand, the function F is translating to the right with velocity

$$v_{group} = \left. \frac{d\omega}{dk} \right|_{k=k_0} .$$

This is the velocity of the whole wavepacket, known as the GROUP VELOCITY. The group velocity is usually the more important physical velocity, since it determines how fast the energy in the wave packet propagates, or in quantum mechanics, how quickly the expectation value for the position of a particle moves.

There is one other important effect that occurs when the wave velocity depends on wavelength, that comes by considering the next order (quadratic) term in the approximation (2). If we had included this term, we would find that rather than simply propagating to the right with the group velocity, the shape of the function F that defines the wavefunction changes with time, generally spreading out. This effect is known as DISPERSION. The reason it happens is that the group velocity also depends on wavelength, and since there are a range of wavelengths contributing to the wavepacket, there are a range of group velocities and this leads to the spreading. Since the range of wavelengths is larger for narrow wavepackets, the spreading occurs faster for these.