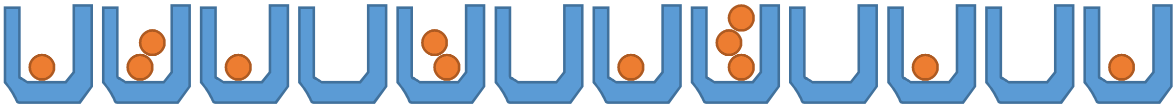


In this assignment, we will consider a simple physical system with M "molecules" each of which can carry units of energy. We will determine the entropy and temperature of the system as a function of energy.

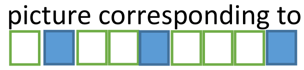
Question 1

For a system with 4 molecules, determine the number of configurations with 0, 1, 2, and 3 units of energy. Write these down explicitly: for example, (0,2,0,1) would be one configuration with 3 units energy. *Note: we are considering the units of energy to be identical, so the only thing we need to keep track of is how many units of energy each molecule has.*

Question 2

For the general case of M molecules, determine (in terms of M) how many configurations there are with 0, 1, and 2 units of energy. Explain your answer. Check your formula by comparing the answers for M=4 with your answers for question 1.

Question 3

Now let's work out the general formula for E units of energy. We can use a trick. For any way of dividing up the energy, we can make a picture with E unfilled boxes and M-1 filled boxes. For example, (1,2,0,3) is associated with the picture  , where the unfilled boxes represent 6 units energy and the 3 filled boxes are dividers, which separate the energy into 4 groups, of size 1, 2, 0, and 3 respectively. To make sure you understand this, draw the picture corresponding to (2,0,3,0) and figure out what distribution (a,b,c,d) corresponds to the picture 

We have learned that for any way of dividing E units of energy between M molecules, we can associate a picture with E + M - 1 boxes, M - 1 of which are filled. So the number of ways of distributing the energy is equal to the number of such pictures. How many is this?

Mathematical aside: the number of ways to choose n objects from a total of N objects is equal to

$$\frac{N!}{n!\,(N-n)!}$$

To see this, notice that there are N choices for the first object, (N-1) choices for the second, and so forth, until (N-n+1) choices for the nth object. This would give N(N-1)…(N-n+1) total ways. But we have overcounted, since for example, choosing A then B is gives us the same two objects as choosing B then A. So we have to divide by n!, the number of possible orderings of n objects. The final answer is

$$\frac{(N-1)\ldots(N-n+1)}{n!} = \frac{N!}{n!\,(N-n)!}$$

This is often written as $\binom{N}{n}$ *and said as "N choose n".*

Question 4

Now we have a formula for N(E) for our system of M molecules. According to our definition of entropy, the entropy of the system is $S = k_B \ln(N(E))$. To get a simple approximate formula for S, we can use a famous formula for logarithms of factorials, that $\ln(A!) \approx A\, \ln(A) - A$ when A is large. Using this formula (and formulae for logarithms of products and quotients), show that when E and M are large, we have approximately that

$$S(E) \approx k_B\, (\, (M+E-1) \ln(M+E-1) - (M-1) \ln(M-1) - E \ln(E))$$

Sketch the entropy as a function of energy (you can pick some value of M and then plot this using a computer).

Question 5

Using your results from section 4 and the definition $T^{-1} = dS/dE$, determine the temperature of the system as a function of energy. Sketch this function.

Question 6

For very small x, we have $\ln(1 + x) \approx x$. Using this, come up with a formula for the temperature in terms of energy that is valid in the case where $E >> M$. Assuming that M is large, does your result agree with our previous idea that temperature should be related to the amount of energy per molecule?

Question 7

In the same limit, $E >> M$, determine the molar specific heat of our system of molecules. Express your answer in terms of R.