PHYS 503 **Problem set # 1 solution** **Jan. 25, 2022**

1 (5 points)

For time-independent Hamiltonians, the time evolution of operators in the Heisenberg picture is given by

$$u_j(t) = e^{iH(t-t_0)} u_j e^{-iH(t-t_0)}$$

where t_0 is some arbitrary initial time. For simplicity, we only consider the case $t > t'$. We can write

$$\begin{aligned} G_{ij}(t,t') &= -i\langle T\left[u_i(t)u_j(t')\right]\rangle = -i\,\langle 0|\, e^{iH(t-t_0)}u_i(t_0)e^{-iH(t-t_0)}e^{iH(t'-t_0)}u_j(t_0)e^{-iH(t'-t_0)}\,|0\rangle \\ &= -i\,\langle 0|\, e^{iE_0(t-t_0)}u_i(t_0)e^{-iH(t-t')}u_j(t_0)e^{-iE_0(t'-t_0)}\,|0\rangle \\ &= -ie^{iE_0(t-t')}\,\langle 0|\, u_i(t_0)e^{-iH(t-t')}u_j(t_0)\,|0\rangle \\ &= G_{ij}(t-t') \end{aligned}$$

In going to the second line we used the fact that $|0\rangle$ is an eigenstate of H with eigenvalue E_0. Also note that any dependence on t_0 drops out.

2 (10 points)

Since the Green's functions depend only on the difference $t - t'$ in our case, we might as well set $t' = 0$:

$$G^{0,R/A}_{ij}(t) = \mp i\theta(\pm t)\langle [u_i(t), u_j(0)]\rangle_0 \tag{1}$$

We saw that, by expressing the displacement operator $u(t)$ in terms of creation and annihilation operators

$$u_i(t) = \frac{1}{\sqrt{2M\Omega_0}}(B_i(t) + B_i^\dagger(t)) \tag{2}$$

the Hamiltonian assumes the quadratic form

$$H_0 = \sum_i \Omega_0 \left(B_i^\dagger B_i + \frac{1}{2}\right)$$

where B_i are second-quantized bosonic operators satisfying the usual commutation relations

$$\left[B_i, B_j^\dagger\right] = \delta_{ij}\,, \quad [B_i, B_j] = \left[B_i^\dagger, B_j^\dagger\right] = 0$$

and

$$B_i(t) = B_i e^{-i\Omega_0 t}\,, \quad B_i^\dagger(t) = B_i^\dagger e^{i\Omega_0 t}\,. \tag{3}$$

Inserting (2) into (1) we get

$$\begin{aligned} G^{0,R/A}_{ij}(t) &= \mp\frac{1}{2M\Omega_0} i\theta(\pm t)\langle\left[e^{-i\Omega_0}tB_i + e^{i\Omega_0 t}B_i^\dagger, B_j + B_j^\dagger\right]\rangle \\ &= \mp\frac{1}{2M\Omega_0} i\theta(\pm t)\left(e^{-i\Omega_0 t} - e^{i\Omega_0 t}\right) \\ &= \mp\frac{1}{M\Omega_0}\theta(\pm t)\sin\Omega_0 t\,. \end{aligned}$$

3 (20 points)

The derivation of the time-ordered, advanced, and retarded Green's functions follows is done in Doniach and Sondheimer. Now, however, we can explicitly state the form of D_{ij}:

$$D_{ij} = -\frac{K}{M}(\delta_{j,i+1} + \delta_{j,i-1})\,.$$

Note that Eq. (1.1.7) in Doniach and Sondheimer assumes that $D_{ij} = D_{ji}$ is symmetric, so that we have made sure to symmetrize above expression. Its Fourier-transform is

$$D_k = \sum_j D_{ij} e^{-ik(R_i - R_j)} = -\frac{K}{M} 2\cos ka\,,$$

where a is the lattice constant. Note that $\Omega_0 = \sqrt{\frac{2K}{M}}$. Then

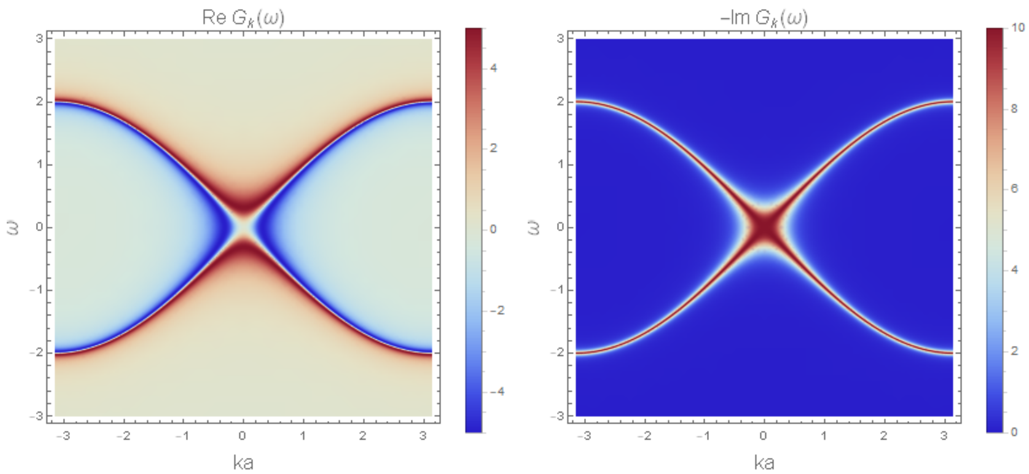
$$\Omega_k^2 = \Omega_0^2 + D_k = \frac{2K}{M}(1 - \cos ka) = \frac{4K}{M}\sin^2\frac{ka}{2}\,.$$

We observe that $\Omega_{k=0} = 0$ and $\Omega_k \simeq c|k|$ for small k as expected for acoustic phonon modes in a solid.

Inserting this expression into the form of the Green's functions derived in Doniach and Sonheimer, we get

$$\begin{aligned}
G_k(\omega) &= \frac{1}{M\left(\omega^2 - \frac{4K}{M}\sin^2\frac{ka}{2} + i\eta\right)} \\
G_k^R(\omega) &= \frac{1}{M\left(\omega^2 - \frac{4K}{M}\sin^2\frac{ka}{2} + i\eta\omega\right)} \\
G_k^A(\omega) &= \frac{1}{M\left(\omega^2 - \frac{4K}{M}\sin^2\frac{ka}{2} - i\eta\omega\right)}
\end{aligned}$$

Just for fun, we can plot the real and imaginary part of the time ordered correlation function $G_k(\omega)$. For this, we choose $K = M = 1$ and $\eta = 0.1$.



The positive ω branch of $\mathrm{Im}G_k(\omega)$ is related to the spectral function and shows essentially the dispersion relation of the acoustic phonons in this simple one-dimensional elastic solid.