LECTURE 15

SUPERCONDUCTIVITY

- Basic phenomenon: Zero resistivity, Meissner effect
  Cooper istability, BCS wavefunction

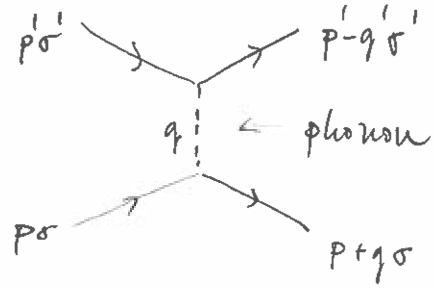
- Consider BCS "pairing Hamiltonian" $H = H_0 + H_1$

$$H_0 = \sum_{k\sigma} \varepsilon_k c^+_{k\sigma} c_{k\sigma} \qquad \varepsilon_k = \frac{k^2}{2m} - \mu$$

$$H_1 = -\frac{1}{2} V \sum_{\substack{pp'q \\ \sigma\sigma'}} c^+_{p+q\sigma} c^+_{p'-q\sigma'} c_{p'\sigma'} c_{p\sigma}$$

$\leftarrow$ attractive interaction (due to phonons)

- We will study a two-electron GF at finite T: ("pair")



$$F(\vec{p}, \vec{p}\,'; \vec{q}; \tau) = \left\langle T\left[c_{p+q\uparrow}(\tau)\, c_{-p\downarrow}(\tau)\, c^+_{p'+q\uparrow}(0)\, c^+_{-p'\downarrow}(0)\right]\right\rangle$$

$$\tau \in (-\beta, \beta) \qquad \beta = \frac{1}{k_B T}$$

We calculate it using FD expansion in terms of unperturbed GF

$$G^0(\vec{p}, \tau) = \left\langle T\left[c_p(\tau)\, c^+_p(0)\right]\right\rangle$$

$$G^0(\vec{p};\tau) = \sum_n e^{i\nu_n \tau} G^0(\vec{p}, \nu_n)$$

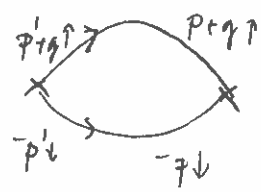
$$G^0(\vec{p}, \nu_n) = \frac{1/\beta}{i\nu_n + \varepsilon_p} \qquad \nu_n = \frac{\pi}{\beta}(2n+1) \qquad n = 0, \pm 1 \dots$$

F is a two-electron GF, therefore its Matsubara frequencies are bosonic

$$F(\vec{p}, \vec{p}'; \vec{q}; \tau) = \sum_m e^{i\omega_m \tau} F(\vec{p}, \vec{p}'; \vec{q}; \omega_m)$$

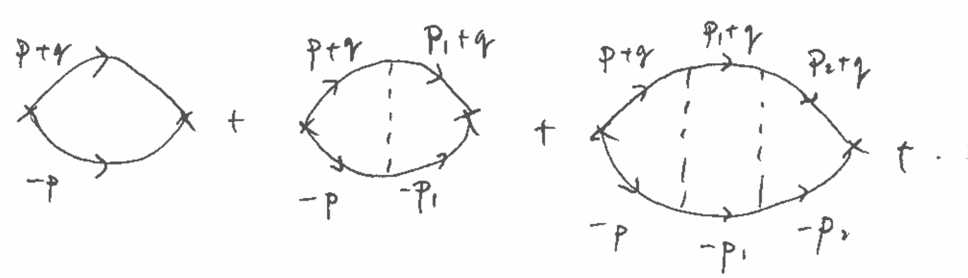
$$\omega_m = (2\pi/\beta) m \qquad m = 0, \pm 1, \dots$$

- Zero-order:

$F^0(\vec{p}, \vec{p}'; \vec{q}; \omega_m) =$ 

$$= -\delta_{pp'} \sum_n G^0(p+q, \omega_m + \nu_n)\, G^0(-p, -\nu_n)$$

- In higher order we consider ladder diagrams that represent processes where electrons in a pair repeatedly scatter from one another:

$F(\vec{p}, \vec{p}'; \vec{q}; \omega_m) =$ 

Because we assumed momentum-indep. interation V in the BCS Hamiltonian the sums over p_1, p_2 separate and one can exactly sum all the diagrams as

$$\mathcal{F}(\vec{q},\omega_m) = \frac{\mathcal{F}^{\circ}(\vec{q},\omega_m)}{1+\beta V \mathcal{F}^{\circ}(\vec{q},\omega_m)}$$

$$\mathcal{F}(\vec{q},\omega_m) = \sum_{p} \mathcal{F}(\vec{p},\vec{p}',\vec{q};\omega_m)$$

- SC instability is signaled by the divergence in pair propagator $\mathcal{F}(\vec{q},\omega_m)$,

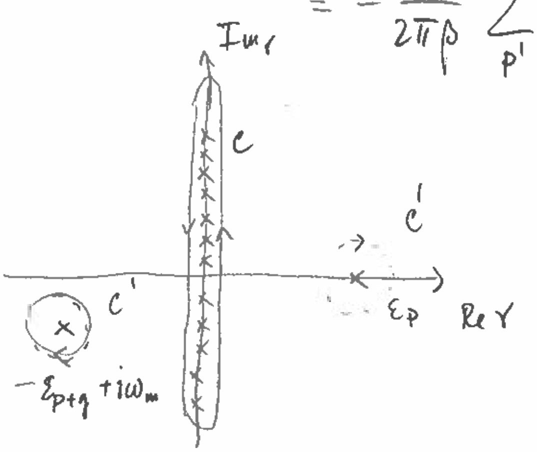
$$1+\beta V \mathcal{F}^{\circ}(\vec{q},\omega_m) = 0$$

$$\mathcal{F}^{\circ}(\vec{q},\omega_m) = -\frac{1}{\beta^2}\sum_{p,n}^{(1)} \frac{1}{(i\omega_m + i\nu_n + \varepsilon_{p+q})}\,\frac{1}{(-i\nu_n+\varepsilon_p)}$$

(1) restriction on $\vec{p}$ summation to $|\varepsilon_p| < k_B\theta_D$

θ_D - Debye temp.

$$= -\frac{i}{2\pi\beta}\sum_{p'} \oint_C \frac{d\nu}{e^{\beta\nu}+1}\,\frac{1}{(\nu+\varepsilon_{p+q}+i\omega_m)(-\nu+\varepsilon_p)}$$



- The largest (negative) value of $\mathcal{F}^{\circ}(\vec{q},\omega_m)$ occurs when $\omega_m = 0 \Rightarrow$ we are interested in $\mathcal{F}^{\circ}(\vec{q},0)$.

$$\mathcal{F}(\vec{q},0) = \frac{1}{\beta}\sum_{p}{}' \left[\frac{1}{(e^{\beta\varepsilon_p}+1)(\varepsilon_p-\varepsilon_{p+q})} - \frac{1}{(e^{-\beta\varepsilon_p}+1)(\varepsilon_p+\varepsilon_{p+q})} \right]$$

$$= -\frac{1}{\beta}\sum_{p}{}' \frac{1-f_{p+q}-f_p}{\varepsilon_{p+q}+\varepsilon_p}$$

$$= -\frac{1}{2\beta}\sum_{p}{}' \frac{\tanh(\frac{1}{2}\beta\varepsilon_{p+q}) + \tanh(\frac{1}{2}\beta\varepsilon_p)}{\varepsilon_{p+q}+\varepsilon_p}$$

- The largest value occurs when $\vec{q}=0$, therefore the denominator is

$$1+\beta V F^{\circ}(0,0) = 1 - V\sum_{p}{}' \frac{\tanh(\frac{1}{2}\beta\varepsilon_p)}{2\varepsilon_p}$$

$$\int_{-k_B\theta_D}^{k_B\theta_D} d\varepsilon\, N(\varepsilon)\, \frac{\tanh(\frac{1}{2}\beta\varepsilon)}{2\varepsilon} \simeq N(0)\int_0^{k_B\theta_D} d\varepsilon\, \frac{\tanh(\frac{1}{2}\beta\varepsilon)}{\varepsilon}$$

$$= N(0)\left[\ln\left(\tfrac{1}{2}\beta k_B\theta_D\right) - \underbrace{\int_0^{\frac{1}{2}\beta k_B\theta_D \to \infty} dx\, \frac{\ln x}{\cosh^2 x}}_{\ln\left(\frac{\pi}{4\gamma}\right)} \right]$$

$\ln\left(\frac{\pi}{4\gamma}\right) \simeq 0.8187$

γ - Euler's const.

$$1 = V N(0) \ln\left(\frac{2\gamma\beta k_B\theta_D}{\pi}\right)$$

$$\Rightarrow \boxed{T_c = 1.14\,\theta_D\, e^{-1/VN(0)}}$$

BCS formula for superconducting critical temperature