LECTURE 12

Frequency - momentum space Feynman diagrams

- most economical way to do perturbation theory
- introduce 4-momenta by combining $\vec{k}$ and ϵ, $k = (\epsilon, \vec{k})$

Then

$$\sum_k G^0(\vec{k}, t=0^-) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\epsilon}{2\pi} G^0(\vec{k}, \epsilon)\, e^{-i\epsilon 0^-}$$

$$= \int \frac{d^4k}{(2\pi)^4} G^0(k)\, e^{-i\epsilon 0^-}$$

This leads to general rules for an nth-order diagram (which contains n vertices and 2n+1 fermionic lines):

(i) The 4-momentum is conserved at each vertex. A factor $V(q)$ is associated with each vertex.

(ii) A factor $G^0(p)$ is associated with each fermion line of 4-momentum p.

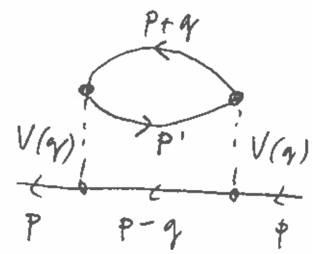
(iii) Integrate over 4-momenta of all <u>internal</u> lines.

(iv) There is a factor (-1) for each closed loop in the diagram and an overall factor $(i)^n$

(v) Consider only topologically distinct diagrams i.e. treat all diagrams as equivalent that are obtained by relabeling variables.

Example: To second order $(n=2)$ there is 10 distinct connected diagrams in $G(p)$ see Eq. (6.3.13)

- One important contribution is



$$(-1)\, i^2\, G^\circ(p) \left[\int \frac{d^4q}{(2\pi)^4} V^2(q)\, G^\circ(p-q) \int \frac{d^4p'}{(4\pi)^2} G^\circ(p')\, G^\circ(p+q) \right] G^\circ(p)$$

Perturbation theory can be simplified by introducing <u>irreducible self energy</u> $\Sigma(p)$

↑ diagrams that cannot be divided in two subdiagrams by cutting a single fermion line.

$\Sigma(p) =$  $+$  $+$  $+$  $+$ ... 4 more at $n=2$

- This again leads to Dyson equation for the propagator

$$G(p) = G^0(p) + G^0(p)\,\Sigma(p)\,G(p)$$

$$\Rightarrow \quad \boxed{G(p) = \frac{G^0(p)}{1 - \Sigma(p)\,G^0(p)}}$$

Recall:

$$G^0(p,\varepsilon) = \frac{1-f_p}{\varepsilon - \varepsilon_p + i\eta} + \frac{f_p}{\varepsilon - \varepsilon_p - i\eta}$$

$$f_p = \frac{1}{e^{\beta\varepsilon_p}+1} \underset{\beta\to\infty}{=} \begin{cases} 1 & |\vec{p}| < p_F \\ 0 & |\vec{p}| > p_F \end{cases}$$

For $|\vec{p}| > p_F$

$$\boxed{G(\vec{p},\varepsilon) = \frac{1}{\varepsilon - \varepsilon_p - \Sigma(\vec{p},\varepsilon) + i\eta}}$$

- <u>Evaluate $\Sigma(p)$ to first order:</u>  $+$ 

(1) (2)

1) $$-i V(0) \sum_k G^0(\vec{k}, t=0^-) = -V(0) \sum_k \lim_{t\to 0^+} \langle T[c_k(-t)\, c_k^+(0)] \rangle$$

$$= V(0) \sum_k \langle c_k^+ c_k \rangle = V(0) \sum_k f_k$$

2) $$i \sum_k V(k-p)\, G^0(k, t=0^-) = -\sum_k V(k-p)\, f_k$$

$$G(\vec{p}, \varepsilon) = \frac{1}{\varepsilon - \bar{\varepsilon}_p + i\delta} \qquad (\text{for } |\vec{p}| > p_F)$$

where $\bar{\varepsilon}_p = \varepsilon_p + V(0) \sum_k f_k - \sum_k V(k-p) f_k$

But this is just the Hartree-Fock result.

- Notice that within HF self-energy is purely REAL, which implies NO DAMPING. Recall

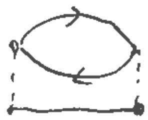
$$G(\vec{p}, \varepsilon) = \frac{1}{\varepsilon - \bar{\varepsilon}_p + i\Gamma_p} \longrightarrow G(\vec{p}, t) \sim -i\, e^{-i\bar{\varepsilon}_p t} \underbrace{e^{-\Gamma_p t}}_{\text{damping}}$$

→ within HF approximation electron excitations have <u>infinite lifetime</u>.

$$\boxed{\text{Q: Is this true beyond HF ?}}$$

Fermi liquid theory (Landau, 1956)

- $n>1$ diagrams produce damping (imaginary part to Σ)
- All qualitative features can be seen from analyzing



this $\overset{n=2}{\text{diagram}}$ (but it is VERY CHALLENGING to prove that this is indeed so)

- Here are some key results of the FL theory: (we now consider $G^R(\vec{k},\varepsilon)$ at non-zero T):

(I) The self energy satisfies

$$\mathrm{Im}\,\Sigma^R(\vec{k},\varepsilon) \sim -(\varepsilon^2+\pi^2T^2)$$

So, at $T=0$ and at low energy $\varepsilon \to 0$ there is NO DAMPING. $\Rightarrow$ Interacting electron system has SHARP QUASIPARTICLES at the Fermi level which do not decay. Scattering rate near FS is (at $T=0$)

$$\frac{1}{\tau_k} \sim -\mathrm{Im}\,\Sigma^R(\vec{k},\bar{\varepsilon}_k) \sim \bar{\varepsilon}_k^2 \to 0 \text{ as } |\vec{k}| \to k_F$$

Note $\bar{\varepsilon}_k$ is defined here such that $\bar{\varepsilon}_{k_F}=0$.

II. Second fundamental property

$$\frac{\partial}{\partial \varepsilon} \operatorname{Re} \Sigma^R(\vec{k},\varepsilon)\Big|_{\varepsilon=\bar{\varepsilon}_k} \le 0$$

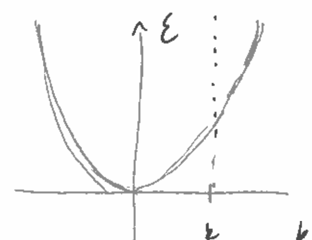
This implies that the "quasiparticle weight"

$$z_k = \frac{1}{1-\partial_\varepsilon \operatorname{Re}\Sigma^R(\vec{k},\varepsilon)}\Big|_{\varepsilon=\bar{\varepsilon}_k}$$

is finite, namely $0 < z_k \le 1$.

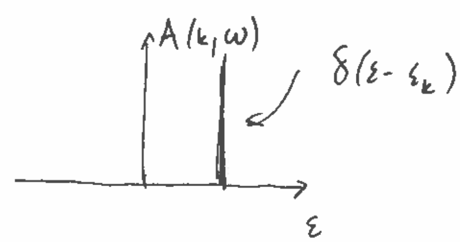
Discussion: Consider $G_0^R(\vec{k},\varepsilon) = \dfrac{1}{\varepsilon-\varepsilon_k+i\eta}$

Spectral function $A_0(\vec{k},\varepsilon) = -\frac{1}{\pi}\operatorname{Im} G_R^0 = \delta(\varepsilon-\varepsilon_k)$



- In HF approximation we'd find that

$$A_{HF}(\vec{k},\varepsilon) = \delta(\varepsilon-\bar{\varepsilon}_k)$$



- In interacting Fermi liquid

$$G^R(\vec{k},\varepsilon) = \frac{1}{\varepsilon+i\eta-\varepsilon_k-\Sigma^R(\vec{k},\varepsilon)}$$

✓ want to understand the structure of this expression

$$A(\vec{k},\varepsilon) = -\frac{1}{\pi}\frac{\operatorname{Im}\Sigma^R(\vec{k},\varepsilon)}{\left(\varepsilon-\varepsilon_k-\operatorname{Re}\Sigma^R(\vec{k},\varepsilon)\right)^2+\left(\operatorname{Im}\Sigma^R(\vec{k},\varepsilon)\right)^2}$$

If $\operatorname{Im}\Sigma^R$ varies weakly for $\varepsilon \approx \varepsilon_k$ then the position of the peak is defined by

$$\varepsilon - \varepsilon_k - \operatorname{Re}\Sigma^R(\vec{k}, \varepsilon) = 0$$

this is true for $\varepsilon = \bar{\varepsilon}_k$

To solve, expand $\Sigma^R(\vec{k}, \varepsilon)$ about $\bar{\varepsilon}_k$:

$$\varepsilon - \varepsilon_k - \operatorname{Re}\Sigma^R(\vec{k}, \varepsilon) \simeq (\varepsilon - \varepsilon_k) - \operatorname{Re}\Sigma^R(\vec{k}, \bar{\varepsilon}_k) - (\varepsilon - \bar{\varepsilon}_k)\,\partial_\varepsilon \operatorname{Re}\Sigma^R\Big|_{\bar{\varepsilon}_k}$$

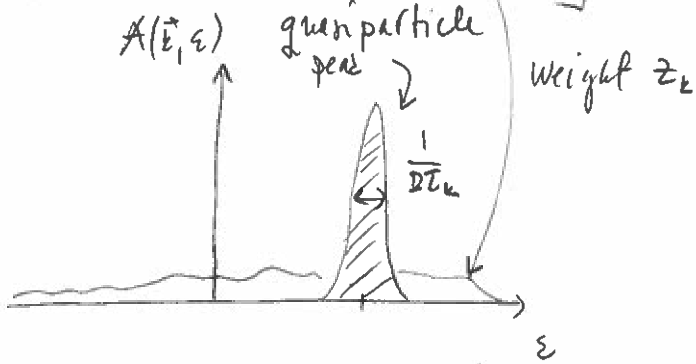
$$= \frac{\varepsilon - \bar{\varepsilon}_k}{Z_k}$$

Therefore,

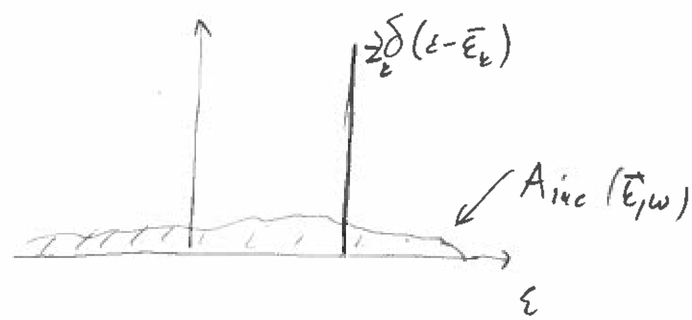
$$A(\vec{k}, \varepsilon) = \frac{Z_k}{\pi} \frac{1/2\tau_k}{(\varepsilon - \bar{\varepsilon}_k)^2 + (1/2\tau_k)^2} + A_{inc}(\vec{k}, \omega)$$

"incoherent part"

where $$\frac{1}{\tau_k} = -2 Z_k \operatorname{Im}\Sigma^R(\vec{k}, \bar{\varepsilon}_k)$$

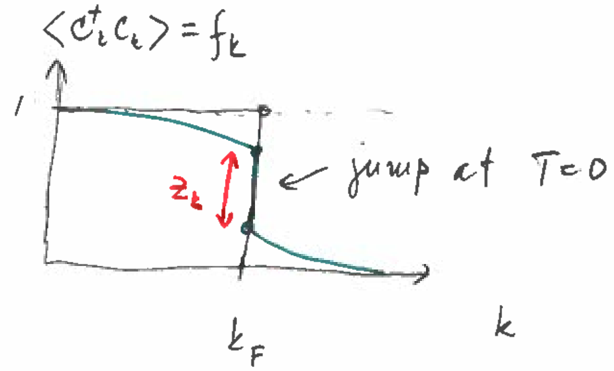


- According to the first property $\frac{1}{2\tau_k} \to 0$ as $\varepsilon \to \bar{\varepsilon}_k$ so we have



- Momentum distribution across the FS

- For non-interacting electrons we have unit jump at FS.



- For interacting FL the jump is still present but its size is reduced to $z_k < 1$.

$$\langle c_k^\dagger c_k \rangle_\beta = G(\vec{k}, \tau = 0^+) = \frac{1}{\beta} \sum_n e^{i\omega_n \tau} G(\vec{k}, \omega_n)$$

$$= \frac{1}{\beta} \sum_n e^{i\omega_n \tau} \int_{-\infty}^{\infty} d\varepsilon \, \frac{A(\vec{k}, \varepsilon)}{i\omega_n - \varepsilon}$$ ← spectral representation

$$= \int_{-\infty}^{\infty} d\varepsilon \, n_F(\varepsilon) A(k, \varepsilon) \xrightarrow{T\to 0} \int_{-\infty}^{0} d\varepsilon \, A(\vec{k}, \varepsilon)$$

Evaluate the jump at $k = k_F$

$$\left(\lim_{k \to k_F^+} - \lim_{k \to k_F^-} \right) \langle c_k^\dagger c_k \rangle = \left(\lim_{k \to k_F^+} - \lim_{k \to k_F^-} \right) \int_{-\infty}^{0} d\varepsilon \, \underbrace{A(\vec{k}, \varepsilon)}_{\xrightarrow{T=0} z_{k_F} \delta(\varepsilon - \bar{\varepsilon}_k)}$$

$$= - z_k$$