

**1. (15 points)** In this assignment we will explore a model of unconventional superconductivity that arises in a system of *spinless* fermions on the square lattice in two dimensions. Consider the following Hubbard Hamiltonian

$$\mathcal{H} = \sum_{ij} (t_{ij} - \mu \delta_{ij}) c_i^\dagger c_j - V_1 \sum_{\langle ij \rangle} n_i n_j \quad (1)$$

for spinless fermions  $c_j$ . Here  $\langle ij \rangle$  denotes summation over nearest neighbor sites  $i$  and  $j$  on the square lattice and  $V_1 > 0$  defines the attractive interaction.

a) By decoupling the interaction term in the pairing channel derive the mean-field Bogoliubov-de Gennes Hamiltonian for this problem, assuming a spatially uniform superconducting state. Note that in this case there will be two pairing fields,  $\Delta_x$  and  $\Delta_y$ , associated with the horizontal and vertical bonds, respectively.

b) Find the Nambu-Gorkov Green's function in the Matsubara frequency/momentum space and use it to derive gap equations for  $\Delta_x$  and  $\Delta_y$  at a non-zero temperature  $T$ .

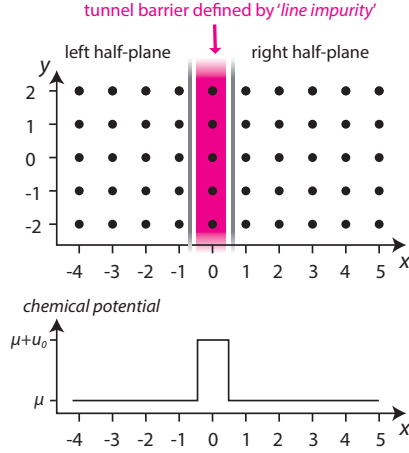
c) Because of the square symmetry of the problem it is reasonable to assume that  $\Delta_x$  and  $\Delta_y$  have the same amplitude but may differ in their phase, that is  $\Delta_x = \Delta$  and  $\Delta_y = e^{i\varphi} \Delta$  with  $\Delta$  real positive. Show that the superconducting state is generically gapless when  $\varphi = 0$  and is gapped when  $\varphi = \pi/2$ . Hint: Consider the character of the excitation spectrum given by  $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ . Gapless means there are excitations at arbitrarily small energies  $E_{\mathbf{k}}$ .

**2. (10+10 points)** For  $\varphi = \pi/2$  the model in Problem 1 describes the so called  $p_x + ip_y$  superconductor which is a canonical example of the topological phase. When placed on a long strip the system will exhibit a fully gapped bulk and topologically protected chiral edge modes (chiral means that they propagate in opposite directions on opposite edges of the strip).

The Green's function perturbation technique described below allows for an elegant approach for visualizing the edge mode spectra. To introduce an edge, we take the Hamiltonian  $\mathcal{H} = \mathcal{H}_0 + \mathcal{U}$  where  $\mathcal{H}_0$  is the mean-field BdG Hamiltonian derived in part 1 and

$$\mathcal{U} = u_0 \sum_{\mathbf{r}} \delta_{x,0} c_{\mathbf{r}}^\dagger c_{\mathbf{r}} \quad (2)$$

is a perturbation. The lattice sites are now labelled by vector  $\mathbf{r} = (x, y)$ . We see that the perturbation introduces a potential barrier along the line  $x = 0$  as illustrated in the figure below. If the barrier is sufficiently high, i.e.  $u_0 \gg t_{\langle ij \rangle}$ , tunnelling between the two infinite half planes  $x > 0$  and  $x < 0$  will be completely suppressed and they become essentially decoupled. In effect, the perturbation creates two independent semi-infinite half planes of a



$p_x + ip_y$  superconductor with boundaries at  $x = \pm 1$ . (Here, the lattice constant is  $a = 1$ .) In practice, one can even take the limit  $u_0 \rightarrow \infty$  at the end of the calculation.

a) Following the calculation for a metal with a single impurity discussed in class (Doniach & Sondheimer ch. 4), derive the perturbation series for the function  $F(\mathbf{p}, \mathbf{p}', \varepsilon)$  defined in Eq. (4.3.9) of the textbook. You will need to generalize the calculation from the textbook, where  $F$  is a scalar, to the case where  $F$  is a  $2 \times 2$  matrix. Next, use your result to obtain an expression for the Nambu-Gorkov Green's function

$$G(x, p_y, \varepsilon) \equiv F(xp_y; xp_y, \varepsilon) = \sum_{p_x p'_x} e^{ip_x x} e^{-ip'_x x} F(p_x p_y; p'_x p_y, \varepsilon). \quad (3)$$

This is a mixed representation where we use the real space coordinate along  $x$ , but momentum coordinate along  $y$ , since the model is translationally invariant in the  $y$ -direction. Note: You can express your result in terms of the general non-interacting Green's function  $G_0$  corresponding to  $\mathcal{H}_0$ . Only for the numerical evaluation in part (b) you need to consider the explicit form of  $G_0$ .

b) [bonus part, 10 points] Evaluate numerically and plot the spectral function  $A(x, p_y, \varepsilon) = -\frac{1}{\pi} \text{Tr} [G(x, p_y, \varepsilon)]$  associated with  $G(x, p_y, \varepsilon)$  as a function of  $p_y, \varepsilon$ . When  $x$  is chosen far away from the boundary this should show the fully gapped bulk spectrum. To observe the edge physics set  $x = 1$  or  $x = -1$ ; this will exhibit the gapless edge modes traversing the bulk gap. An example of such a spectral function (computed for a  $d_{x^2-y^2} + id_{xy}$  superconductor) is shown in the figure below. Use the model parameters  $\Delta = 2t$  and  $\mu = -2t$  where  $t$  denotes the nearest-neighbor tunneling amplitude and all longer range hoppings are taken to be zero. What is the difference between  $x = 1$  and  $x = -1$ ?

