1. (10 points) In the previous problem set we derived an expression for the phonon self-energy in the Holstein model

$$\Pi_q(\omega_n) = -2M\Lambda^2\beta\Omega_q \sum_k \frac{f_k - f_{k-q}}{i\omega_n + \varepsilon_k - \varepsilon_{k-q}}, \tag{1}$$

valid to second order in e-ph coupling Λ.

a) By performing the analytic continuation and using the Dyson's equation find the expression for the retarded phonon propagator $D_q^R(\omega)$ that follows from the above result.

b) Show that the real part of $\Pi_q^R(\omega)$ can be viewed as the leading correction to the phonon dispersion Ω_q due to coupling to the electron sea.

c) Evaluate this correction explicitly for the free electron gas in one dimension with $\epsilon_k = k^2/2m$ at $T = 0$. Show that the correction is singular when $q \to \pm 2k_F$ and discuss the physical significance of this singularity.

2. (10 points) Section 6.2 of the textbook shows how to calculate the exchange contribution to the ground state energy of the interacting electron gas in Hartree-Fock approximation.

a) Use this result to write down an expression for the total ground state energy (i.e. including the trivial non-interacting part) in terms of r_s and a_0.

b) The above result assumes that the numbers $N_\pm$ of spin up and down electrons are the same. More generally one can consider a *polarized* electron gas in which $N_+ \neq N_-$. Find the ground-state energy of such a polarized system in the HF approximation as a function of total number $N = N_+ + N_-$ of electrons and the polarization $M = (N_+ - N_-)/N$.

c) Show that ferromagnetic state ($M = 1$) has a lower energy than the unmagnetized state ($M = 0$) if r_s exceeds critical value r_s^c. Explain why this is so. Find r_s^c.

3. Bonus problem (5 points) For an electron moving in the potential of randomly distributed short-range impurities suppose you wanted to find the propagator that includes an infinite series of 'nested' diagrams indicated in the figure. How would you accomplish this?